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Markov chains of nonlinear Markov processes and an application to a winner-takes-all model for social conformity

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Abstract
We discuss nonlinear Markov processes defined on discrete time points and discrete state spaces using Markov chains. In this context, special attention is paid to the distinction between linear and nonlinear Markov processes. We illustrate that the Chapman–Kolmogorov equation holds for nonlinear Markov processes by a winner-takes-all model for social conformity.

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Nonlinear Markov processes were introduced a few decades ago [1] and since then have been extensively studied in the context of nonlinear Fokker–Planck equations [2–19]. However, the concept of a nonlinear Markov process is a fundamental one and applies to all kinds of Markov processes. In order to demonstrate this point, we will discuss nonlinear Markov processes defined on discrete time points and discrete state spaces using Markov chains. We first present a definition of nonlinear Markov processes in terms of discrete time Markov chains and dwell on implications of this definition. In particular, we will see that transition probabilities of nonlinear Markov processes depend on occupation probabilities of states. Since the evolution of these occupation probabilities in turn is determined by transition probabilities, nonlinear Markov processes exhibit a circular causality structure that is tailored to address the stochastic phenomenon in self-organizing systems. In several studies it has been advocated that such self-organizing processes are at the heart of social dynamics and can be used to address the issues of social group behavior in general and collective opinion formation and social conformity in particular [20–24]. In line with these studies, we will illustrate the Markov property itself by means of an example in the field of social psychology: a winner-takes-all model for social conformity.

Let i denote a state of a state space Ω that corresponds to a countable set of integers (e.g. $\Omega = \{1, \ldots, 10\}$). Let us use the variables $X_n^{(k)} \in \Omega$ for $n = 1, 2, \ldots$ and $k = 1, \ldots, N$ with $N \to \infty$ to describe trajectories (or realizations) of a discrete time stochastic process. Then

$p_i(n)$ denotes the probability of finding realizations in the state i at time step n. The stochastic process under consideration is assumed to satisfy the iterative map

$$p_j(n+1) = \sum_{i\in\Omega} T_{i\to j}(\{p_l(n)\}_{l\in\Omega})\, p_i(n), \tag{1}$$

where $T_{i\to j}$ denotes the conditional probability that a realization being in the state i jumps to the state j. Equation (1) has to be solved for initial distributions given in terms of the probabilities $p_i(1)$. If the conditional probabilities $T_{i\to j}$ do not depend on the probability distribution $\{p_l(n)\}_{l\in\Omega}$, then equation (1) is linear with respect to the process probabilities. Stochastic processes in this case are referred to as linear Markov processes. If there is at least one matrix element $T_{i\to j}$ that depends on the distribution $\{p_l(n)\}_{l\in\Omega}$, then equation (1) is nonlinear with respect to $\{p_l\}_{l\in\Omega}$. By analogy with the time continuous case [2, 3, 9, 18, 19], we will refer in this case to stochastic processes as nonlinear Markov processes. In order to illustrate the Markov property of processes defined by equation (1) we use a simplified notation that holds for finite spaces Ω of the form $\Omega = \{1,\ldots,L\}$, probability vectors $\mathbf{p}$ defined by $\mathbf{p} = (p_1,\ldots,p_L)$ and $L\times L$ matrices $\hat{T}$ composed of the conditional probabilities $T_{i\to j}$. Using this notation, equation (1) reads

$$\mathbf{p}(n+1) = \hat{T}(\mathbf{p}(n))\cdot\mathbf{p}(n). \tag{2}$$

Iteration of equation (2) yields

$$\mathbf{p}(n+2) = \hat{T}(\mathbf{p}(n+1))\cdot\hat{T}(\mathbf{p}(n))\cdot\mathbf{p}(n), \tag{3}$$

which can equivalently be expressed as

$$\mathbf{p}(n+2) = \hat{T}(\hat{T}(\mathbf{p}(n))\cdot\mathbf{p}(n))\cdot\hat{T}(\mathbf{p}(n))\cdot\mathbf{p}(n). \tag{4}$$

In order to evaluate higher iterations let us introduce the $L\times L$ matrices

$$\begin{aligned}
\hat{T}_1(\mathbf{p}) &= \hat{T}(\mathbf{p}),\\
\hat{T}_2(\mathbf{p}) &= \hat{T}(\hat{T}(\mathbf{p})\cdot\mathbf{p}),\\
\hat{T}_3(\mathbf{p}) &= \hat{T}(\hat{T}(\hat{T}(\mathbf{p})\cdot\mathbf{p})),\\
&\ldots\\
\hat{T}_m(\mathbf{p}) &= \underbrace{\hat{T}(\hat{T}(\cdots(\hat{T}(\mathbf{p})\cdot\mathbf{p})\cdots)}_{\hat{T}\text{ occurs } m \text{ times}}.
\end{aligned} \tag{5}$$

Consequently, we have $\mathbf{p}(n+m) = \hat{T}_m(\mathbf{p}(n))\cdot\hat{T}_{m-1}(\mathbf{p}(n))\cdots\hat{T}_1(\mathbf{p}(n))\cdot\mathbf{p}(n)$, which can be written as

$$\mathbf{p}(n+m) = \hat{T}^{(m)}(\mathbf{p}(n))\cdot\mathbf{p}(n), \tag{6}$$

using the m-step conditional probability $L\times L$ matrix

$$\hat{T}^{(m)}(\mathbf{p}) = \hat{T}_m(\mathbf{p})\cdot\hat{T}_{m-1}(\mathbf{p})\cdots\hat{T}_1(\mathbf{p}), \tag{7}$$

where $\mathbf{p}$ corresponds to $\mathbf{p}(n)$ in all terms. The m-step conditional probability matrix satisfies

$$\hat{T}^{(m+r)}(\mathbf{p}(n)) = \hat{T}^{(r)}(\mathbf{p}(n+m))\cdot\hat{T}^{(m)}(\mathbf{p}(n)), \tag{8}$$

which is the Chapman–Kolmogorov equation for nonlinear Markov processes [18, 19]. Note that equation (8) can alternatively be expressed by

$$\hat{T}^{(m+r)}(\mathbf{p}) = \hat{T}^{(r)}(\hat{T}^{(m)}(\mathbf{p})\cdot\mathbf{p})\cdot\hat{T}^{(m)}(\mathbf{p}), \tag{9}$$

where $\mathbf{p}$ corresponds to $\mathbf{p}(n)$ in all terms.

Let us illustrate the Markov property as expressed by the Chapman–Kolmogorov equation (8) for a winner-takes-all model of social conformity. Conformity is a fundamental phenomenon in the field of group behavior. People tend to have attitudes, opinions and behavior patterns that are in line with generally accepted attitudes, opinions and behavioral patterns. In other words, individuals tend to align their thinking and performance with standards defined by groups [25, 26]. In this context, it does not matter whether or not the group opinion or behavior is correct in a objective sense. For example, in a seminal study by Asch, subjects of a group had to match a so-called 'standard line' with three other lines labeled 1, 2, 3. Only line 2 had the exact length of the 'standard line'. Lines 1 and 3 were different in length and this difference was obvious. In Asch's experiment, all subjects of the group but one were 'fake subjects' (or actors) that gave (by purpose) wrong answers and explained that the 'standard line' matched line 3. Asch found in many cases that the 'real subject' adopted the group behavior and matched the 'standard line' with line 3 as well [25, 26]. In what follows, we discuss a winner-takes-all model that can be used to describe the emergence of social conformity. We consider a state space $\Omega = [a, \ldots, b]$ with $b > a$ that represents attitudes, opinions and behavioral patterns between two extremes represented by the states a and b. For example, the states i may describe how strong individuals agree or disagree with a particular statement or idea. In particular, we may think of students of a class that were asked to discuss whether the next exam should be a homework exam or an exam written in class. Alternatively, the states i may represent to which extent individuals participate in a particular trend. In particular, the trend could be the possession and utilization of cell phones, where the state b describes individuals carrying their cell phones all the time and the state a describes individuals that do not want to possess a cell phone at all.

In what follows, we will describe individuals in terms of realizations of a stochastic process defined by equation (1). In order to define the conditional probabilities $T_{i\to j}$ of equation (1), we will distinguish between the intrinsic dynamics of individuals and the interaction dynamics of our model. The intrinsic dynamics describes the individual behavior in the absence of a group. We model the intrinsic dynamics by a random walk and, in doing so, neglect individual preferences. The interaction dynamics accounts for interactions between groups and individuals. We assume that if the concentration of individuals at a state i^* is larger than a threshold θ, then a group has been established at that state i^* that acts as an absorbing entity. Consequently, individuals in adjacent states are attracted to the state i^* and individuals being in the state i^* will not be able to leave the group state i^*. We define concentration in terms of the probability p_i^* of realizations (or individuals) being in the state i^*. In detail, our model involves the following conditional probabilities $T_{i\to j}$. For the intrinsic dynamics on $[a+1, \ldots, b-1]$ we have

$$\left.\begin{array}{lll} T_{i\to j}(\{p_l\}_{l\in\Omega}) = 0.5 & \text{for} & j = i \pm 1 \\ T_{i\to j}(\{p_l\}_{l\in\Omega}) = 0 & & \text{otherwise} \end{array}\right\} i \neq a; i \neq b, \quad p_i < \theta. \tag{10}$$

For the intrinsic dynamics on the boundary states a and b we have

$$T_{a\to a+1} = T_{a\to a} = T_{b\to b-1} = T_{b\to b} = 0.5. \tag{11}$$

For the interaction dynamics between groups and individuals we have

$$\left.\begin{array}{lll} T_{i\to j}(\{p_l\}_{l\in\Omega}) = 0 & \text{for} & j \neq i \\ T_{i\to i}(\{p_l\}_{l\in\Omega}) = 1 & & \end{array}\right\} i \neq a; i \neq b, \quad p_i \geqslant \theta. \tag{12}$$

Finally, we require that $\theta > 1/(b - a + 1)$ and $p_i(1) > 0$ for all $i \in \Omega$. That is, the threshold concentration θ must be larger than the single-state probabilities of the uniform distribution and there should be no unoccupied states in the initial condition. The stochastic process defined by

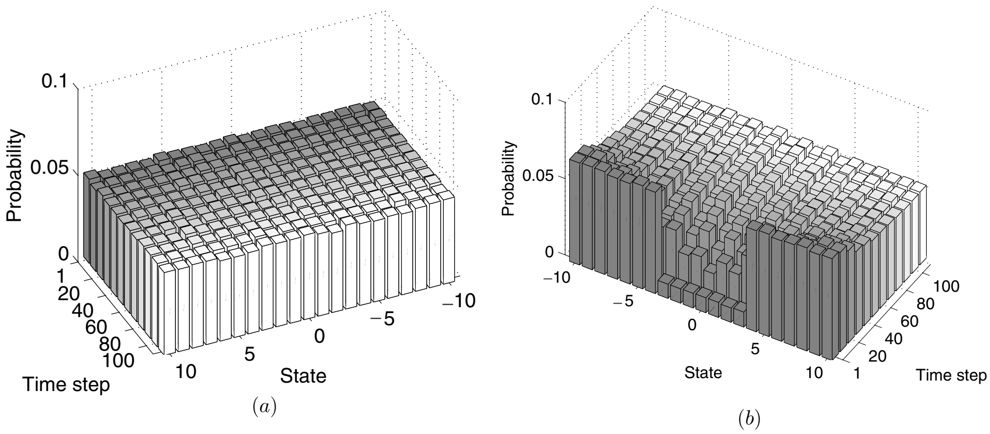
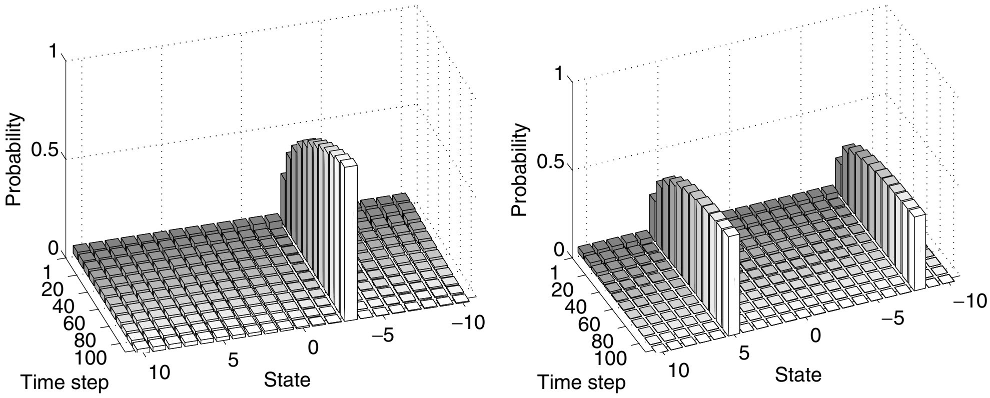


Figure 1. Evolution of the probability distributions $p_i(n)$ with $p_i(1) < \theta$ for all states i. Panel (*a*): initial uniform distribution. Panel (*b*): initial distribution with a gap in the spectrum of opinions or behavioral patterns at $i = -3, \ldots, 3$.

equations (1) and (10)–(12) exhibits several possible stationary distributions $\{p_{l,\mathrm{st}}\}_{l\in\Omega}$. First of all, the uniform distribution with $p_{i,\mathrm{st}} = 1/(b-a+1)$ for all i is a stationary distribution. The reason for this is that if the initial probabilities $p_i(1)$ are all smaller than θ, then at every time step n we are dealing with an ordinary random walk. The maximal probability $p_{\max}$ of the probabilities $p_i(n)$ is constant or decreases at every time step and the interaction dynamics (12) can be neglected because $p_i(n) < \theta$ holds for all $n \geqslant 1$. In short, if $p_i(1) < \theta$ holds for all i, then the stochastic process converges to a uniform distribution and corresponds to a random walk restricted to the domain Ω. Simulations of this case are shown in figure 1.

Next, let us consider an initial distribution that involves a subset of probabilities that are larger than or equal to θ. Let $W = \{i \mid p_i(1) \geqslant \theta\}$ denote the corresponding index set (the 'winner states'). Let $L = \{i \mid p_i(1) < \theta\}$ denote the complementary index set (the 'loser states'). Note that these states cannot surpass the threshold θ in subsequent time steps. That is, they cannot become 'winner states'. The reason for this is that the probability of a state i at step n cannot exceed the average of the probabilities of its adjacent states $i \pm 1$ at step $n-1$. If the adjacent states at step $n-1$ are occupied with probabilities smaller than θ, then the state i at step n is occupied with a probability smaller than the threshold value θ. If one of the adjacent states has a probability larger than the threshold θ at step $n-1$, then there are no transitions from this adjacent state to the state i and the probability of state i at time step n is simply half of the probability of the adjacent state with the subthreshold probability. Therefore, from $p_i(1) < \theta$ for $i \in L$ it follows that $p_i(n) < \theta$ holds for all $n \geqslant 1$ and $i \in L$. In general, if there is at least one winner state, then every realization X_n of the stochastic process under consideration that occupy states $i \in L$ performs a random walk restricted to a certain interval $(a', b') \subset [a, b]$ with $a' < b'$, where a' or b' correspond to absorbing boundaries (i.e. winner states $a', b' \in W$) or to boundary states (i.e. $a' = a$ or $b' = b$). In any case, at least one of the interval boundaries a' and b' corresponds to an absorbing boundary in terms of a 'winner state'. The implications are twofold. First, the maximum probability $p_{\max}(n)$ of the set of probabilities $p_i(n)$ with $i \in L$ is constant or decreases as a function of n. Second, the probability

$$P_L(n) = \sum_{i\in L} p_i(n) \tag{13}$$



(a) (b)

Figure 2. Panel (a): evolution of the probability distribution $p_i(n)$ for an initial distribution involving one single-state i^* with $p_{i^*}(1) > \theta$ (here $i^* = -3$). Panel (b): as in panel (a) but for an initial distribution involving two states i^* with $p_{i^*}(1) > \theta$ (here $i_1^* = -8$ and $i_2^* = 5$).

decreases monotonically as a function of n and $P_L(n)$ converges to zero because at every time step n there are transitions from the 'loser states' to the 'winner states' (recall that we required above $p_i(1) > 0\ \forall i$). Turning to the 'winner states', we see that from $p_i(1) \geqslant \theta$ for $i \in W$ it follows that $p_i(n) \geqslant \theta$ holds for all $n \geqslant 1$ and $i \in W$. Moreover, from $P_L(n) \to 0$ for $n \to \infty$, it follows that

$$P_W(n) = \sum_{i \in W} p_i(n) \to 1 \tag{14}$$

holds in the limiting case $n \to \infty$. Since jumps between states $i \in W$ are forbidden, the stochastic process converges to a process where all realizations are trapped in 'winner states' and the probabilities p_i for $i \in W$ become stationary. This implies that the nonlinear Markov process as a whole becomes stationary. The corresponding stationary distributions are characterized by

$$\begin{aligned} p_{i,\mathrm{st}} &> 0 \qquad \text{for} \quad i \in W, \\ p_{i,\mathrm{st}} &= 0 \qquad \text{for} \quad i \in L. \end{aligned} \tag{15}$$

The numeric values of the probabilities $p_{i,\mathrm{st}}$ related to the 'winner states' depend on the initial distribution $\{p_l(1)\}_{l \in \Omega}$. We see that the 'winner states' finally capture all realizations, which is the reason why we may refer to our model as a winner-takes-all model. Figure 2 illustrate the results derived so far.

With respect to our application to social conformity, we see that our model describes the growth of groups with common attitudes, opinions, believes and behavioral patterns. The specific attitude or behavioral pattern that is represented by a group is not pre-defined. A group at a state i is established if the initial concentration is sufficiently large irrespective of what attitude, opinion, believe or behavioral pattern is represented by that state i. This is consistent with our note made previously regarding Asch's seminal experiment. Social conformity is related to the fact that there is an impressive concentration of people who think and act in the same way and is not related to a particular standard that is represented by a group.

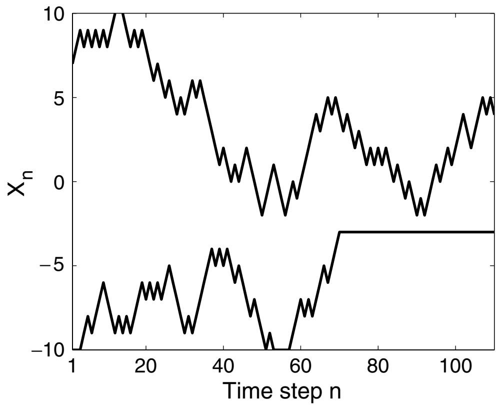


Figure 3. Two realizations of the Markov process with the probability distribution shown in panel (*a*) of figure 2. The realization shown in the lower part gets trapped in the state i^* with $p_{i^*}(1) > \theta$. The realization shown in the upper part is not trapped at time step $n = 110$ but will become a trapped realization as well in the limit $n \to \infty$.

Figure 3 highlights that equation (1) can be used to compute stochastic trajectories. In fact, we computed the probability distributions shown in figures 1 and 2 from trajectories as shown in figure 3. To this end, we simulated an ensemble of $N = 100\,000$ realizations for $\theta = 0.1, a = -10$ and $b = 10$. For the simulation shown in figure 1 we distributed in the initial time step $n = 1$ all realizations uniformly across Ω (panel (*a*)). In panel (*b*) of figure 1 we distributed again realizations uniformly across Ω but we re-distributed subsequently 80% of the realizations of the states $-3, -2, -1, 0, 1, 2, 3$ to states with $i < -3$ or $i > 3$. In doing so, we obtained an initial distribution with $p_i(1) < \theta$ for all i that exhibits a gap in the spectrum of opinions or behavioral patterns represented by the states i. For the simulation shown in panel (*a*) of figure 2 we constructed the initial distribution by preparing 20% of all realizations in one state ($i = -3$) and distributing the remaining 80% uniformly across Ω. For the simulation shown in panel (*b*) of figure 2 we constructed the initial distribution by preparing 15% of all realization in state $i = -8$, 20% in state $i = 5$ and distributing the remaining realizations uniformly across Ω. Using a Monte Carlo simulation we iterated the ensemble of realizations. That is, we changed the states $X_n^{(k)} \in \Omega$ of the realizations $k = 1, \ldots, N$ by $-1, 0, +1$ at every time step $n = 1, 2, \ldots$ according to the conditional probabilities (10), (11) and (12). We also used the trajectories $X_n^{(k)}$ to illustrate that the Chapman–Kolmogorov equation (8) holds. To this end, we focused on the example shown in panel (*a*) of figure 2 that involves only one single 'winner state'. We computed from $X_n^{(k)}$ the conditional probabilities

$$T_{0\to k}^{(20)}(\{p_l(1)\}_{l\in\Omega}), \qquad T_{k\to 0}^{(20)}(\{p_l(21)\}_{l\in\Omega}), \qquad T_{k\to w}^{(20)}(\{p_l(21)\}_{l\in\Omega}), \qquad (16)$$

where w denotes the index of the 'winner state' (i.e. $w = -3$). These conditional probabilities are shown in panels (*a*), (*b*) and (*c*) of figure 4. Note that in fact we repeated the Monte Carlo simulation shown in figure 2(*a*) for the same initial probability distribution several times (20 repetitions). The graphs shown in figure 4 report the mean values and standard deviations (in terms of error bars) thus obtained. Furthermore, we computed (for all simulation repetitions) the conditional probabilities $T_{0\to 0}^{(41)}(\{p_l(1)\}_{l\in\Omega})$ and $T_{0\to w}^{(41)}(\{p_l(1)\}_{l\in\Omega})$ directly from the trajectories $X_n^{(k)}$ and, in addition, from the Chapman–Kolmogorov equation (8) like

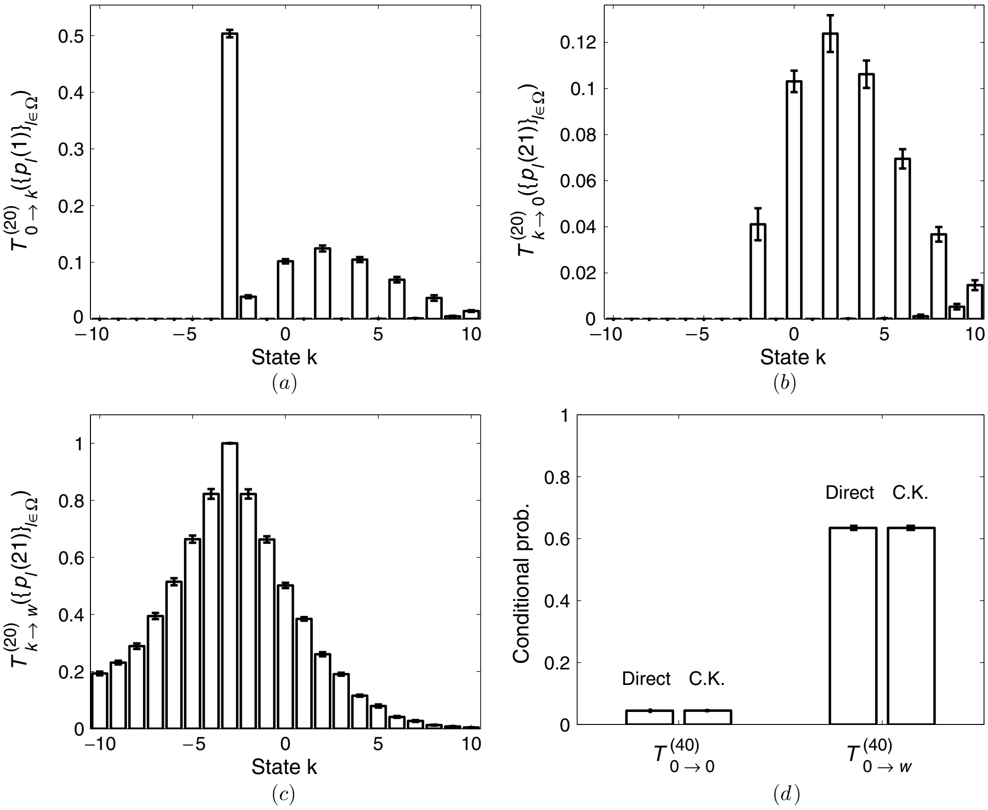


Figure 4. Illustration of the Chapman–Kolmogorov equation (8). Panels (*a*)–(*c*) depict the conditional probabilities $T^{(20)}_{0\to k}(\{p_l(1)\}_{l\in\Omega})$, $T^{(20)}_{k\to 0}(\{p_l(21)\}_{l\in\Omega})$ and $T^{(20)}_{k\to w}(\{p_l(21)\}_{l\in\Omega})$ of the nonlinear Markov processes shown in figure 2(*a*). Panel (*d*) summarizes the conditional probabilities $T^{(40)}_{0\to 0}(\{p_l(1)\}_{l\in\Omega})$ and $T^{(40)}_{0\to w}(\{p_l(1)\}_{l\in\Omega})$ calculated directly from the numeric data ('direct') or computed by means of equation (17) ('C.K.') and the distributions shown in panels (*a*)–(*c*). The results shown in panels (*a*)–(*d*) are estimates (averages) obtained from a set of 20 Monte Carlo simulations. Error bars represent standard deviations of these estimates. See text for details.

$$
\begin{aligned}
T^{(41),\mathrm{C.K.}}_{0\to 0}(\{p_l(1)\}_{l\in\Omega}) &= \sum_{k\in\Omega} T^{(20)}_{k\to 0}(\{p_l(21)\}_{l\in\Omega}) T^{(20)}_{0\to k}(\{p_l(1)\}_{l\in\Omega}),\\
T^{(41),\mathrm{C.K.}}_{0\to w}(\{p_l(1)\}_{l\in\Omega}) &= \sum_{k\in\Omega} T^{(20)}_{k\to w}(\{p_l(21)\}_{l\in\Omega}) T^{(20)}_{0\to k}(\{p_l(1)\}_{l\in\Omega}).
\end{aligned}
\tag{17}
$$

The conditional probabilities $T^{(41)}_{0\to 0}(\{p_l(1)\}_{l\in\Omega})$ and $T^{(41)}_{0\to w}(\{p_l(1)\}_{l\in\Omega})$ computed directly from the trajectories $X^{(k)}_n$ and the conditional probabilities $T^{(41),\mathrm{C.K.}}_{0\to 0}(\{p_l(1)\}_{l\in\Omega})$ and $T^{(41),\mathrm{C.K.}}_{0\to w}(\{p_l(1)\}_{l\in\Omega})$ obtained from the Chapman–Kolmogorov equation (8) are depicted in panel (*d*) of figure 4. In line with the theoretical considerations made above, we found a good match between these quantities. Note that the standard deviations for the four probability estimates reported in panel (*d*) were smaller than 0.01 which is the reason why the corresponding error bars are hardly visible.

In conclusion we would like to address two issues: a fundamental and an applied one. One main objective of our study was to introduce stochastic processes described by equation (1) as Markov processes. We demonstrated the Markov property by showing that the Chapman–Kolmogorov equation (8) holds. Actually, implicitly we demonstrated even more rigorously the Markov property by performing Monte Carlo simulations of equation (1). At every time step n the Monte Carlo simulation requires as input variables an ensemble of realizations $X_n^{(k)}$. It does not require information about the realizations $X_m^{(k)}$ at prior time steps $m < n$. That is, stochastic processes described by Markov chains of the form (1) are completely defined for $n \geqslant n^*$ provided that the probability distribution $\{p_l\}_{l\in\Omega}$ at n^* is known. This is the reason why processes defined by equation (1) are Markov processes. The more applied objective of our study was to highlight that nonlinear Markov processes are tailored to address group behavior. Interactions between groups and individuals can be modeled in terms of interactions between statistical ensembles and realizations. The proposed model for the emergence of social conformity accounts for such interactions between groups and individuals. However, our proposed model neglects various details. The interaction between groups and individuals features just two modes: there is a trapping effect or there is no trapping effect depending on whether or not the concentration of an opinion or behavioral pattern exceeds a critical threshold θ. Therefore, our model may be generalized to account for interactions that depend in a more gradual fashion on the distribution of opinions and behavioral patterns. Likewise, we neglected preferences of individuals by assuming that the intrinsic dynamics of individuals is given by a simple random walk. Again, our model could be generalized to account for preferences by replacing the unbiased random walk by a stochastic process that involves attractors and repellers. Such generalizations would be helpful to match model predictions to empirical data. However, such generalization would not affect the key property of our proposed model: it admits for multiple stationary distributions that reflect the fundamental observation that groups can emerge due to group pressure irrespective of what kind of opinion, believe, attitude or behavioral pattern is represented by a group.

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